MULTILATERAL INDEX NUMBER SYSTEMS FOR INTERNATIONAL PRICE COMPARISONS: PROPERTIES, EXISTENCE AND UNIQUENESS

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Abstract

Over the last five decades a number of multilateral index number systems have been proposed in the context of spatial and cross-country price comparisons. These indexes have been devised and refined for the purpose of compiling purchasing power parities of currencies in the International Comparison Program at the World Bank. These multilateral index numbers are usually expressed as solutions to systems of linear or nonlinear equations. In this paper, we establish the necessary and sufficient conditions for existence and uniqueness of solutions for a host of known index number systems (including the Geary-Khamis, Ikle, Neary and Rao index) as well as some new systems. A feature of the reported results is that the condition for existence of solutions can be stated in terms of an easily verifiable condition based on observed quantities of different commodities in different countries. The theorems proved are likely to be relevant for any new index number system that might be considered in the future and makes use of the twin concepts of PPPs and international average prices.

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1. Introduction

Bilateral index numbers such as the Laspeyres, Paasche, Fisher, and Tornqvist are used by economists and statisticians around the world in the compilation of consumer price index numbers and gross domestic product deflators. These index numbers are essentially binary in nature and are designed to measure changes in prices over two periods. Price comparisons over several periods use chained indexes linked through binary price comparisons over adjacent time periods. Over the past fifty years, a new strand of index number theory has emerged in response to the need for index numbers that can be used for multilateral price and quantity comparisons. Multilateral price comparisons involving more than two periods or spatial entities[1] involve simultaneous comparisons of all possible pairs of countries or periods involved. These are expected to satisfy a host of properties or axioms such as *transitivity, base invariance, characteristicity, and additivity.* Diewert (1988, 1999) and Balk (2009) provide an overview of the axiomatic or test properties of multilateral index number systems. The need for specially designed systems arises due to the fact that most of the commonly used index numbers such as the Laspeyres, Paasche, Fisher and Tornqvist indices do not satisfy transitivity.

Over the last 50 years there have also been increasing demands for international price comparisons and for the compilation of internationally comparable macroeconomic aggregates such as the gross domestic product (GDP), private consumption, gross fixed capital formation and government expenditure. Typically cross-country price comparisons involve multilateral comparisons of prices across all pairs of countries. These price comparisons are usually referred to as purchasing power parities of currencies (PPPs) which are now regularly compiled as a part of the International Comparison Program (ICP)[2]. The

[1] Spatial entities may refer to countries, regions or cities within countries, or any other suitably identified geographical region. Throughout this paper, we use countries to facilitate presentation and discussion.
[2] Since the inception of the International Comparison Program (ICP) in 1968 and the start of regular compilation of purchasing power parities (PPPs), under the auspices of the UN Statistical Commission (UNSC) and undertaken by international organization such as the World Bank, OECD, Eurostat, the Asian and African

most recently released findings from the ICP are for the year 2011 covering 199 countries of the world (World Bank, 2014). The PPPs from the ICP are used in assessing the size and distribution of the World Economy and the rankings of economies by the real size rather than through nominal measures based on exchange rates. The ICP 2011 report indicates that the United States was the largest in 2011 followed by China, India and Japan. Results from the ICP also indicate that there has been a significant reduction in global inequality based on PPP-converted per capita data obtained from the ICP.

How are the PPPs compiled? The PPPs within the ICP are obtained by aggregating price data collected from different countries using an appropriate multilateral index formula[3]. A variety of multilateral index numbers have been proposed for the purpose of PPP compilation over the last five decades. The most notable are the *Gini-Elteto-Koves-Szulc* (GEKS); Geary-Khamis (GK); generalized GK; Iklé; Rao; and weighted *Country-Product-Dummy* (WCPD) methods. Rao (2013b) and Diewert (2013) describe the methods currently employed within the ICP. Hill (1997) provides a taxonomy of multilateral index number systems. In addition there are also spatial chaining methods based on *minimum spanning trees* e.g. Hill, (1999, 2009); Neary (2004) and Feenstra, Ma and Rao (2009). Rao (2009) offers a collection of papers that describe the state of the art and advances made in this direction. Balk (2009) provides a technical overview of the state of the art in terms of index number systems used in international comparisons. Despite these, the quest for indexes with better properties still continues and new index are being proposed e.g. see Hajargasht and Rao, (2010) and Rao and Hajargasht, (2014) for a new stochastic approach, and Hill, (2009) and Rao, Shankar and Hajargasht, (2010) for spatial chaining approach to the compilation of PPPs.

A common attribute of the multilateral index number systems used in international comparisons is that the price indexes from these methods are often obtained as solutions to some suitably defined systems of equations. These systems can be linear as in the case of the GK and Iklé systems or nonlinear as is the case with the Neary (2004) and Rao (1976) systems. These systems can be meaningful only if solutions which are positive and unique (up to a factor of proportionality[4]) exist. Therefore considerable efforts have been put into proving the existence of solutions to these systems (e.g., Geary-Khamis, 1972; Rao, 1971 and 1976; Neary, 2004) but the results are not always complete and satisfactory. The main

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Development Banks, there has been an increased interest on the use of multilateral index number systems in the compilation of PPPs of currencies and also spatial price index numbers.

[3] Details of the ICP methodology can be found in Rao (2013a) and World Bank (2013).

[4] A constant multiple of a given set of PPPs leaves price comparisons between countries unchanged. Therefore, it is sufficient if PPPs are determined uniquely up to a factor of proportionality.

objective of this paper is to develop a technical toolkit in the form of a set of general theorems that can be used in establishing existence and uniqueness of solutions to most known and potential new multilateral index number systems. We also introduce the *axiom of units of reference currency unit* and show that it leads to a new class of multilateral index number systems based on generalized means of order ρ. This new class of multilateral system encompasses most of the known systems such as Geary-Khamis, Iklé and Rao systems. The paper also provides a set of easily verifiable conditions that guarantee the existence of solutions to these index numbers.

In this paper we prove three general theorems on existence and uniqueness of such indexes and offer some easily verifiable necessary and sufficient conditions under which solutions to some of the widely used index number systems exist. In theorem one, we consider a general system that can be turned into a linear system. We prove necessary and sufficient conditions for existence and uniqueness of this system. We show that the two popular indexes of Geary (1958)-Khamis (1972), Iklé (1976) as well as a few other indexes are special cases of the system considered in this theorem. The second general theorem is nonlinear and has Neary (2004) and Rao (1976) systems as its special case. We prove existence of this class of index numbers and state conditions under which uniqueness is guaranteed. In our third theorem, we prove sufficient conditions for existence and uniqueness of solutions for another system which encompasses some of the popular index numbers. An interesting feature of the theorems established in the paper is that the necessary and sufficient conditions can be translated into simple conditions on the observed quantities. In particular our results show that in general existence and uniqueness is guaranteed if the observed quantity matrix over commodities and countries is *connected*. We also demonstrate that the general systems discussed in these theorems can be used as a basis for generating new index number systems.

The paper is organized as follows. In section 2 we establish the basic notation and concepts that underpin the multilateral index number systems considered in this paper. This section establishes a general framework for solving these index number systems and states the nonlinear eigenvalue theorem which is in turn used in proving theorems stated in Section 4. Section 3 describes linear/nonlinear systems that characterize several commonly used systems of index numbers for international comparisons. Section 4 states and prove the main theorems on existence and uniqueness of general classes of multilateral index numbers. The general theorems are also used to prove the existence and uniqueness of many of the index numbers currently used in international comparisons as well as in generating some new multilateral index number systems. The paper concludes with some general remarks in Section 5.

2. Notation, basic concepts and preliminaries

We begin this section by introducing the basic notation and the concepts that underpin the multilateral systems for the compilation of PPPs for international price comparisons. Let p_{ij} and q_{ij} represent the price and the quantity of the *i*-th commodity in the *j*-th country respectively where $j = 1, \ldots, M$ and $i = 1, \ldots, N$. We assume that all the prices are positive and all the quantities are non-negative. Without loss of generality, we assume: (i) for each *i*, q_{ij} is positive for at least one *j* ; and (ii) for each *j*, q_{ij} is positive for at least one *i*.

Purchasing Power Parities and International Average Prices

Let PPP_j denote the purchasing power parity of currency of country *j* or the general price level in *j*-th country relative to a numeraire country. PPP_j shows the number of currency units of country *j* that have the same purchasing power, with respect to a basket of goods and services, as one unit of currency of a reference country. For example, if PPP for currency of India is equal to INR 15.50 with respect to one US dollar then 15.50 Indian rupees in India have the same purchasing power as one US dollar in the United States. The multilateral price comparisons consist of the matrix of all binary comparisons. The bilateral price comparisons between two countries *j* and *k*, denoted by P_{jk}, are given by

$$P_{jk} = \frac{PPP_k}{PPP_j} \quad for\ all\ \ j\, and\, k \qquad (1)$$

It is easy to see from (1) that the set of all binary comparisons, P_{jk} (*j,k = 1,2,…,M),* are transitive[5] and that the price comparisons are unchanged if all the PPPs are multiplied by a non-zero constant. Note that for multilateral price comparisons to be meaningful, it is necessary that PPP_j s are strictly positive and determined uniquely up to a factor of proportionality so that P_{jk} in (1) is unique. An implication of (1) is that any multilateral

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[5] Transitivity simply requires that all the pairwise comparisons are internally consistent and satisfy the following condition: $P_{jk} = P_{jl} \cdot P_{lk} \ \ \forall \ j,k$ and l. Rao and Banerjee (1985) proved that a matrix of binary comparisons satisfies transitivity if and only if there exist a set of numbers which can be denoted as $PPP_j \ (j = 1,2,..,M)$ such that $P_{jk} = \dfrac{PPP_k}{PPP_j} \ \ for\ all\ \ j\, and\, k \cdot$

index number system that leads to PPPs must be positive and need to be determined uniquely up to a factor of proportionality.

We also note here that PPP_j would naturally be functions of observed price and quantity data. A common feature of the index number systems we study in this paper is that these systems also determine international average prices of the commodities included in the comparisons. Let P_i denotes the world average price of *i*-th commodity (i =1,2,…,N). Typically these P_i s are also expressed as functions of the observed price and quantity data as well as PPP_j s of currencies discussed above. Maintaining symmetry, PPP_j s are also defined in terms of observed price and quantity data as well as international prices. Intuitively, international average price of a commodity is essentially an averages of prices of the commodity observed in different countries. But we note here that prices are denominated in different currencies in different countries. Therefore, prior to averaging prices across countries these need to be converted into a common currency unit. This is achieved by converting observed prices by PPPs of currencies. In the sections below, we provide different methods of averaging prices such as arithmetic, geometric or harmonic averages with or without weights.

As additional notation, we let $\mathbf{p}$ and $\mathbf{q}$ represent $(N \times M)$ matrices of prices and quantities of all commodities in all the countries. These matrices are observed where prices are all assumed strictly positive whereas the quantity matrix $\mathbf{q}$ has elements which are non-negative but each row and each column of $\mathbf{q}$ have at least one element which is strictly positive.[6]

Multilateral Index Number System

Definition 1: A multilateral index number system is an interrelated system of equations which express the unknown purchasing power parities and international prices as functions of price and quantity data from different countries. Typically, in addition to being functions of observed price and quantity data, each PPP_j is a function of all the international prices and similarly each P_i is a function of all the unknown PPP_j s thus leading to a system of $(M+N)$ equations in M unknown PPP_j s and N unknown P_i s. A general multilateral system may be specified as a system of equations of the following form:

[6] This is the same as the assumption that each commodity is consumed in at least one country and each country consumes at least one commodity.

$$\begin{cases} P_i = H_i^1(\mathbf{PPP}, \mathbf{p}, \mathbf{q}) & (i = 1, \ldots.., N) \quad (2.a) \\ PPP_j = H_j^2(\mathbf{P}, \mathbf{p}, \mathbf{q}) & (j = 1, \ldots.., M) \quad (2.b) \end{cases}$$

where **PPP** and **P** are, respectively, (M × 1) and (N ×1) vectors of purchasing power parities and world average prices; The functions H_i^1s and H_j^2 are strictly positive and continuous in all the arguments. H_i^1 is often in the form of some weighted average of p_{ij}/PPP_j s and H_j^2 is in the form of some weighted average of p_{ij}/P_i s . Different index number systems differ in their specification of functional forms for equations (2.a and 2.b).

Equation system (2.a, 2.b) represents a typical multilateral index number system used in international comparisons. If the main focus is on purchasing power parities, we can substitute equation (2.a) into (2.b) which then results in a homogeneous system of M equations of the form:

$$\begin{aligned} PPP_j &= H_j^2\left(\left[H_i^1\left(\mathbf{PPP}, \mathbf{p}, \mathbf{q}\right) : i = 1, 2, \ldots, N\right], \mathbf{p}, \mathbf{q}\right) \\ &= G_j(\mathbf{PPP}, \mathbf{p}, \mathbf{q}) \qquad\qquad j = 1, 2, \ldots, M \end{aligned} \qquad (3)$$

This means that the set of equations in (3) are such that each PPP_j is expressed as a function of observed price and quantity data as well as the other PPP_j s. As both H_i^1s and H_j^2 are strictly positive, the resulting function ***G*** is also strictly positive. Solving equations in (3) is equivalent to solving the following system of linear homogeneous functions:

$$x_j = G_j(x_1, x_2, \ldots, x_M) \qquad j = 1, 2, \ldots, M$$

or equivalently in matrix form as:

$$\mathbf{x} = \mathbf{G}(\mathbf{x}) \quad \text{or} \quad \mathbf{x} - \mathbf{G}(\mathbf{x}) = \mathbf{0} \qquad (4)$$

To prove the existence of solutions to (3) or the system in (4) we use some form of (non)linear eigenvalue (or Perron-Frobenius) theorems as the main tool. The following is the most well-known theorem proved in Morishima (1964, pp 195-199) or Nikaido (1968, pp.149-161)[7].

Nonlinear Eigenvalue Theorem: Let $G_j(x_1, x_2, \ldots.., x_M)$ for $j = 1, \ldots.., M$ satisfy the following conditions:

(i) Continuity and Non-negativity: $\mathbf{G}(\mathbf{x})$ is a continuous function $R_{++}^M \to R_{++}^M$

[7] The linear version of this theorem is known as Perron-Frobenius theorem (see e.g. Gantmacher 1959, Chapter 3).

(ii) Homogeniety: functions $G_j(x_1, x_2, ..., x_M)$ for $j = 1, ..., M$ are homogenous of degree one

(iii) Monotonicity: for all $\mathbf{x} \geq \mathbf{y}$, $G_j(\mathbf{x}) \geq G_j(\mathbf{y})$

(iv) Indecomposibility: For any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} \geq \mathbf{y} \geq \mathbf{0}$, define nonempty proper subset $\Omega \subset \{1, ..., M\}$ as $\{j \mid x_j > y_j\}$. Function $\mathbf{G}(\mathbf{x})$ is indecomposable if there exists $k \notin \Omega$ such that $G_k(x_1, x_2, ..., x_M) \neq G_k(y_1, y_2, ..., y_M)$.

Under the above conditions, there is a unique positive $\mathbf{x}^*$ (up to a positive scalar multiple) and λ^* such that

$$G_j(x_1^*, ..., x_M^*) = \lambda^* x_j^* \quad (\forall\ j = 1, ..., M)$$

A few remarks are in order.

(i) The nonlinear eigenvalue theorem plays an important role in establishing conditions for the existence of solutions to the multilateral index number systems examined in this paper. It is sufficient if the two functions, H_i^1 and H_j^2, are such that the function $\boldsymbol{G}$ obtained through (3) satisfies the conditions stated in the nonlinear eigenvalue theorem above.

(ii) In the context of multilateral systems we need to find a solution that satisfies equation (4) i.e. $\mathbf{x} = \mathbf{G}(\mathbf{x})$ holds. For this to hold we need to establish that that $\lambda^* = 1$.

(iii) The critical condition in the theorem is indecomposibility. It ensures that a change in one of the x_i s have an effect on all the other x_i s. In this paper, we provide easily verifiable conditions to check indecomposibility in terms of the quantity data matrix.

(iv) There are a host of other nonlinear eigenvalue theorems which considers the situation where some of these assumptions (in particular this concept of indecomposibility) are not satisfied. In theorem 3, we use one such theorem. See e.g. Lemmens and Nussbaum (2012) or Gaubert and Gunawardena (2004) for reviews of nonlinear eigenvalue theorems.

In the ensuing sections of the paper, we state and prove three general theorems regarding the existence and uniqueness of solutions to different types of multilateral index number functions, and illustrate how these theorems can be used in establishing the existence of a number of well-known index number systems including the Geary-Khamis, Ikle, Neary and Rao systems.

3. Multilateral Index Numbers Systems used in international price comparisons

In this section, we describe several multilateral index number systems used in international price comparisons. The objective here is not to provide the reader with a detailed description and discussion of properties of the systems considered, instead the purpose is simply to set the scene for the general theorems stated and proved in the paper and illustrate how these theorems can be used in establishing the existence and uniqueness of solutions to these systems.

Geary-Khamis (GK) System: This system was first proposed by Geary (1958) as a method for comparing agricultural output across countries and the method was later extended to its general form by Khamis (1972). The index was adopted as the main aggregation method in the International Comparison Program (ICP) until 1985 (see Kravis, Heston and Summers, 1982, for a discussion of the method). The GK system consists of the following system of (*M*+*N*) equations:

$$\begin{cases} \dfrac{1}{PPP_j} = \displaystyle\sum_{i=1}^{N} \frac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}} \frac{P_i}{p_{ij}} & (j=1,.....,M) \\[2ex] P_i = \displaystyle\sum_{j=1}^{M} \frac{q_{ij}}{\sum_{j=1}^{M} q_{ij}} \frac{p_{ij}}{PPP_j} & (i=1,.....,N) \end{cases} \qquad (5)$$

The necessary and sufficient conditions for existence and uniqueness of this index have been derived by Khamis (1972) and Rao (1971), but the proofs offered in these papers are somewhat long and tedious.

We observe here that PPP_j is defined as a weighted harmonic mean of price relatives $\left(p_{ij}/P_i\right)$ which makes comparison of prices in country *j* with international average prices. Similarly the international average price, P_i , is defined as a quantity-share weighted average of prices in different countries after they are converted into a common currency using PPP_j s. This illustrates how equations (2.a) and (2.b) are usually in the form of averages.

Generalized Geary-Khamis Method: Cuthbert (1999) proposed the following generalized class of index number systems in order to show that GK is not unique in possessing additivity property[8].

$$\begin{cases} \dfrac{1}{PPP_j} = \displaystyle\sum_{i=1}^{N} \frac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}} \frac{P_i}{p_{ij}} & (j = 1,.....,M) \\ P_i = \displaystyle\sum_{j=1}^{M} \frac{\beta_j q_{ij}}{\sum_{j=1}^{M} \beta_j q_{ij}} \frac{p_{ij}}{PPP_j} & (i = 1,.....,N) \end{cases} \qquad (6)$$

It can be easily seen that Geary-Khamis system in (5) is a special case of (6) when $\beta_j = 1$ for all $j = 1,2,...,M$. Appropriate choices of β_j s lead to the Iklé system [e.g. Balk (1996), p. 207] and to the system proposed in Sakuma, Rao and Kurabayashi (2009). Cuthbert (1999) does not provide any results on the existence of solutions to the Generalized GK system.

A new Multilateral System based on generalized means: Note that in the above systems, e.g. equations in (5), each PPP_j is defined as a weighted harmonic mean of ratios $\left\{ P_i / p_{ij} : i = 1,2,...,N \right\}$ and each P_i is an arithmetic mean of the ratios $\left\{ p_{ij} / PPP_j : j = 1,2,...,M \right\}$. In this paper we propose a new multilateral system based on generalized means of these ratios which can encompass all the systems discussed thus far. In the system defined below we make use of generalized means of any order $\rho \in R$.

$$\begin{cases} \dfrac{1}{PPP_j} = \left(\displaystyle\sum_{i=1}^{N} w_{ij} \left[\frac{P_i}{p_{ij}} \right]^{\rho} \right)^{1/\rho} & (j = 1,.....,M) \\ P_i = \left(\displaystyle\sum_{j=1}^{M} \frac{\beta_j q_{ij}}{\sum_{j=1}^{M} \beta_j q_{ij}} \left[\frac{p_{ij}}{PPP_j} \right]^{\rho} \right)^{1/\rho} & (i = 1,.....,N) \end{cases} \qquad (7)$$

where w_{ij} is the expenditure share defined as: $w_{ij} = \dfrac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}}$

[8] Cuthbert (1999) disproves a conjecture made in Rao (1997) which surmised that the GJK system is the only multilateral system satisfying additivity. For a definition and discussion of additivity see Kravis, Heston and Summers (1982) and also Sakuma, Rao and Kurabayashi (2009).

A common feature of the systems in (5), (6) and (7) is that they share the same system of weights. The PPP_j definition uses expenditure share weights whereas the international prices are defined using quantity share weights. It is easy to show that the GK system defined above and the Iklé system defined below are special cases of the new system defined in (7). Of particular interest is the case with $\rho \to 0$ where the new system turns to the geometric version of the GK index.

Note that in all of these systems, some function of PPP_j s (e.g. inverse of it) is a weighted average of a function of P_i / p_{ij} s and some function of P_i s (e.g. identity) is a weighted average of p_{ij} / P_i s. Theorem 1 proves existence and uniqueness of the solution for a very general form that encompasses all the above cases where many different forms of weights or averages can be used.

Iklé (1972), Rao (1990) and related Multilateral Index Number Systems: We now turn to multilateral index number systems that make use of weighting systems based on expenditure shares instead of just quantities[9]. In order to define these indexes we need to define expenditure share weights w_{ij} and weights w_{ij}^* as:

$$w_{ij} = \frac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}} \quad \text{and} \quad w_{ij}^* = \frac{w_{ij}}{\sum_{j=1}^{M} w_{ij}} \tag{8}$$

By definition, it can be seen that $\sum_{i=1}^{N} w_{ij} = 1$ and $\sum_{j=1}^{M} w_{ij}^* = 1$.

Rao (1990) defines a system for international price comparisons as follows

$$\begin{cases} PPP_j = \prod_{i=1}^{N} \left(\dfrac{p_{ij}}{P_i} \right)^{w_{ij}} & (j = 1, \ldots\ldots, M) \\ P_i = \prod_{j=1}^{M} \left(\dfrac{p_{ij}}{PPP_j} \right)^{w_{ij}^*} & (i = 1, \ldots\ldots, N) \end{cases} \tag{9}$$

Rao (2005) has shown that this system can be obtained as weighted least squares estimates in the country-product-dummy (CPD) method.[10]

In the system proposed by Iklé (1972) and discussed in Balk (1996) and Dikhanov (1994)[11], expenditure share based weights are used along with harmonic averages shown in (10).

[9] Note that the GK system uses quantity share weights in defining international average prices. In the systems we consider here all the weights are based on expenditure shares.

[10] Details of this and other related results can be found in Rao (2009) and Rao and Hajargasht (2014).

$$\begin{cases} \dfrac{1}{PPP_j} = \sum_{i=1}^{N} \left(w_{ij} \dfrac{P_i}{p_{ij}} \right) & (j = 1, ..., M) \\ \dfrac{1}{P_i} = \sum_{j=1}^{M} \left(w_{ij}^{*} \dfrac{PPP_j}{p_{ij}} \right) & (i = 1, ..., N) \end{cases} \quad (10)$$

Note that in Rao system in (9), PPPs and world or international average prices are defined as geometric means of some national prices converted into a common currency using PPPs (p_{ij} / PPP_j) while in Iklé system in (10) harmonic means of the converted national prices are used in a similar manner. Hajargasht and Rao (2010) proposed a similar system of equations but using arithmetic means:

$$\begin{cases} PPP_j = \sum_{i=1}^{N} \left(w_{ij} \dfrac{p_{ij}}{P_i} \right) & (j = 1, ..., M) \\ P_i = \sum_{j=1}^{M} w_{ij}^{*} \dfrac{p_{ij}}{PPP_j} & (i = 1, ..., N) \end{cases} \quad (11)$$

A new general index number system which encompasses all the above systems can be defined as:

$$\begin{cases} PPP_j = \left\{ \sum_{i=1}^{N} w_{ij} \left(\dfrac{p_{ij}}{P_i} \right)^{\rho} \right\}^{1/\rho} & (j = 1, ..., M) \\ P_i = \left\{ \sum_{j=1}^{M} w_{ij}^{*} \left(\dfrac{p_{ij}}{PPP_j} \right)^{\rho} \right\}^{1/\rho} & (i = 1, ..., N) \end{cases} \quad (12)$$

It is well known that the different values for ρ leads to different indexes (see e.g. Arrow et. al 1961). For example $\rho = 0$ leads to Rao system, $\rho = -1$ gives Iklé and $\rho = 1$ leads to the arithmetic index[12]. Theorem 3 proved in this paper can be used to establish the existence and uniqueness of all of these systems and more.

Neary (2004) and Rao (1976) Systems: Neary (2004) proposed a modified version of GK system which introduces standard consumer theory and the notion of the Konus cost of living

[11] The system described in Iklé (1972) was difficult to follow and was ignored until the work of Dikhanov (1994) and Balk (1996) who provided an alternative formulation that is easy to understand and that connects to other known index number systems. Diewert (2013) refers to this system as Iklé-Balk-Dikhanov (IDB) system but we continue to refer to this system in rest of the paper as Iklé system

[12] If such a system of equations can be associated with an estimable stochastic model we might be able to statistically test between these index numbers.

index number (see Diewert, 1976 for details) into the GK system. The Neary system uses the Konus index to define PPPs by evaluating the expenditure necessary to attain a given utility level at the prices observed in country *j* and at the international average prices without going into too much detail, the Neary (2004) system can be described as:

$$\begin{cases} \dfrac{1}{PPP_j} = \sum_{i=1}^{N} P_i q_{ij}^*(\mathbf{P},\mathbf{p_j},\mathbf{q_j}) \Big/ \sum_{i=1}^{N} p_{ij} q_{ij} & (j=1,.....,M) \\ P_i = \sum_{j=1}^{M} \dfrac{q_{ij}}{\sum_{j=1}^{M} q_{ij}^*(\mathbf{P},\mathbf{p_j},\mathbf{q_j})} \dfrac{p_{ij}}{PPP_j} & (i=1,.....,N) \end{cases} \tag{13}$$

where $q_{ij}^*(\mathbf{P},\mathbf{p_j},\mathbf{q_j})$ in (13) are optimal (cost-minimizing) quantities obtained as the solution to the following cost minimization problem

$$\min \sum_{i=1}^{N} P_i q_{ij}^* \quad \text{subject to} \quad U(\mathbf{q_j^*}) \geq U(\mathbf{q_j})$$

where *U(.)* is a well-behaved utility function[13]. This problem is solved for each country separately.

The Rao (1976) system is similar to the Neary (2004) system except that the international average prices are defined in (13) are modified as follows:

$$P_i = \sum_{j=1}^{M} \frac{q_{ij}}{\sum_{j=1}^{M} q_{ij}} \frac{p_{ij}}{PPP_j} \qquad i = 1, 2, ..., N \tag{14}$$

Proof of existence and uniqueness of solutions for the Rao-system is given in Rao (1976) where it makes use of the nonlinear eigenvalue theorem which is also used in our Theorem 2.

Now we turn to the main contribution of the paper which is in the form of three theorems that can be used in establishing the existence properties of the systems discussed above.

4. General Theorems on Existence and Uniqueness of Multilateral Index Number Systems

In this section we state and prove three general theorems related to specific functional forms of H_j^1 and H_i^2 that define a multilateral index number system. Examples of functional forms that pertain to some of the known multilateral systems and a few new generalized classes of multilateral systems are shown in equations (5) to (14). In order to state and prove these

[13] Further details can be found in Neary (2004) and Rao (1976).

theorems, we need to introduce the notion of connectedness across countries that is directly related to the notion of *"indecomposibility"*.

Definition 2 - Connectedness: The quantity matrix $\mathbf{q}$, an $N \times M$ matrix of quantities, $q_{ij}\ (i = 1, 2, ..., N;\ j = 1, 2, ..., M)$ where q_{ij} is the quantity of *i-th* commodity consumed in *j-th* country, is said to be connected if the set of all countries involved in the international comparisons cannot be split into two or more disjoint subsets such that there are no commodities that are commonly consumed across the two country groups. An equivalent mathematical definition of connectedness is that for every non-empty subset of countries $J \subset \{1, 2, ..., M\} \neq \emptyset$ there exists at least one country $j \in J$, one country $l \notin J$ and one commodity $k \in \{1, ..., N\}$ such that both $q_{kj} > 0$ and $q_{kl} > 0$. Another equivalent mathematical definition is that for every non-empty proper subset $\mathrm{I} \subset \{1, ..., N\}$ and $\mathrm{J} \subset \{1, ..., M\}$ with I^c and J^c as their complements, $\mathbf{q}_{I^c J^c} = \mathbf{0}$ implies $\mathbf{q}_{IJ} \neq \mathbf{0}$ where $\mathbf{q}_{IJ} = \{q_{ij}, i \in \mathrm{I}, j \in \mathrm{J}\}$.

Connectedness of $\mathbf{q}$ matrix is critical in international comparisons. If $\mathbf{q}$ is disconnected then countries can be divided into two groups such that they have no common items of consumption. In such a case, there is no basis for making price comparisons. Thus connectedness of $\mathbf{q}$ can be considered necessary. In this paper, we show that this condition is also necessary and sufficient for the existence and uniqueness of several classes of multilateral index number systems. It is possible to give connectedness a graph theoretic interpretation. We define the graph associated with an observed quantity matrix.

Definition 3- Quantity-adjacent Graph: Let $\mathbf{G_q}$ represent a graph associated with a given quantity matrix $\mathbf{q}$ with countries as vertices of the graph. Two vertices *j* and *k* are connected by an *edge* if there exists a commodity $i \in \{1,2,..., N\}$ such that $q_{ij} > 0$ and $q_{ik} > 0$.

Definition 4- Connected Graph: A graph, $\mathbf{G_q}$ is said to be *connected* if for any pair of countries *j* and *k* there exists a sequence of countries $\{j_1, j_2, ..., j_k\} \subset \{1,2,..., M\}$ such that each consecutive pair of countries in the sequence is connected by an edge.

From definitions 2.1 to 2.3 it is easy to establish that the graph $\mathbf{G_q}$ associated with a quantity matrix $\mathbf{q}$ is connected if and only if the quantity matrix $\mathbf{q}$ is connected (Rao, 1971, 1976).

We introduce a definition that is closely related to the idea of connectedness of $\mathbf{Q}$ and the existence of multilateral index numbers which is the central theme of the paper.

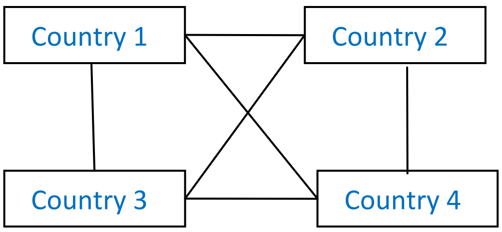
***Definition 5- Indecomposibility[14] of a Square Matrix*:** A non-negative $M \times M$ matrix $\mathbf{A} = \{a_{ij}\}$ is said to be indecomposable if and only if for any proper subset $J \subset \{1, 2, ..., M\} \neq \emptyset$ there exists at least one $j \in J$ and $i \notin J$ such that $a_{ij} > 0$ (see e.g. Nikaido 1968 pp 105).

Indecomposible Matrices and Adjacent Graphs – An illustration

The notions of indecomposibility of the matrix of quantities in different countries and the connectedness of the graphs associated with quantity matrices are central to the existence theorems proved below. The graph theoretic interpretation provides powerful algorithms for checking indecomposibility. We illustrate these notions using three examples where we consider four countries and four commodities. The quantity matrix, **A**, in this case is of order 4 × 4. Without loss of generality we have countries in the rows and commodities in the columns.

The first example is where all the four commodities are consumed in all the countries. As a result, the graph associated with the matrix will have edges connecting all pairs of countries as shown below.

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 100 & 25 \\ 6 & 3 & 75 & 35 \\ 80 & 25 & 250 & 125 \\ 8 & 6 & 35 & 40 \end{bmatrix}$$



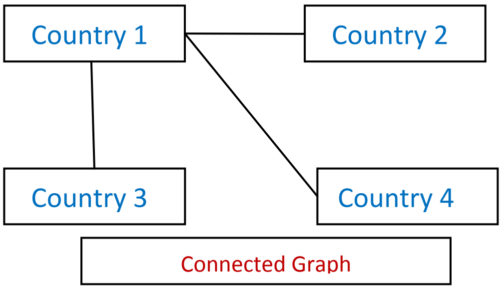
Complete Graph

In this case it is easy to see that the matrix **A** is indecomposable. The adjacent graph is a complete graph where each country is directly connected with every other country. This is a case where there is strong connectedness.

The second example is where the first country consumes all the four commodities whereas the other three countries consume only one or two of the commodities. The quantity matrix and the adjacent graphs in this case are of the following form.

[14] Also known as irreducibility

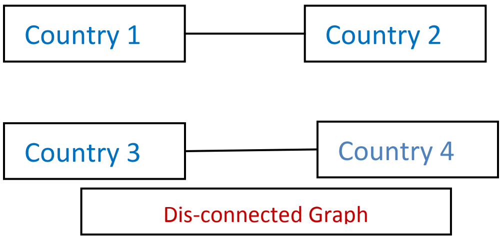
$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 100 & 25 \\ 6 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 35 & 40 \end{bmatrix}$$



It is easy to check that the matrix **A** here is indecomposable. We can see that Country 1 is connected to rest of the countries. However, there are no direct links between countries 2, 3 and 4. In this case the graph is connected but with no cycles – this means between any two countries there is only one chained path connecting the two countries. For example, countries 3 and 2 are connected through country 1. This type of graph is referred to as a spanning tree. In this case there is only weak connectivity through country 1, however this is sufficient for indecomposibility of the matrix **A**.

Finally, we consider the example where the four countries are divided into two groups, countries 1 and 2 and countries 3 and 4. These groups have no commodities commonly consumed. Therefore the matrix **A** is decomposable matrix and it consists of two sets of matrices of lower dimension (two) each of which are indecomposable but the full matrix is not indecomposable.

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ 0 & 0 & 250 & 125 \\ 0 & 0 & 35 & 40 \end{bmatrix}$$



In this example, countries 1 and 2 have commodities (1 and2) commonly consumed whereas countries 3 and 4 consume only commodities 3 and 4. In this case, the adjacent graph is disconnected. There is no connection between either country 3 or country 4 with countries 1 or 2. In this case there is no basis for multilateral comparisons involving all the four countries.

We now state the first theorem that is closely linked with the Geary-Khamis, generalized Geary-Khamis and related systems. In the theorem below the functional forms for H^1 and H^2 that define PPP_j's and P_i's are linear in weights but allow for nonlinear functions of the unknowns.

Theorem 1: *Consider the following general system of (N+M) equations*

$$\begin{cases} f_j\left(PPP_j\right)=\sum_{i=1}^{N} a_{ij}\, g_i(P_i)\big/e_{ij} & j=1,2,...,M \quad (15.a) \\ g_i(P_i)=\sum_{j=1}^{M} b_{ij}\, e_{ij}\, f_j(PPP_j) & i=1,2,...,N \quad (15.b) \end{cases}$$

where $f_j(.)$*s and* $g_i(.)$*s can be any bijective functions from* $R_+ \to R_+$; a_{ij} and b_{ij} are non-negative functions of $(\mathbf{p},\mathbf{q})$ with $a_{ij}=\left(c_{ij}\Big/\sum_{j=1}^{M} c_{ij}\right)$ *and* $b_{ij}=\left(d_{ij}\Big/\sum_{i=1}^{N} d_{ij}\right)$ and $e_{ij}(\mathbf{p},\mathbf{q})>0$ *such that* $c_{ij}=d_{ij}e_{ij}$ *and* $q_{ij}>0 \Leftrightarrow d_{ij}>0$ **.**

Then a necessary and sufficient condition for the existence of a "unique" positive solution for this system is connectedness of countries as in Definition 2.

Before offering a proof for this theorem, we examine various aspects of the theorem.

(i) First, note that as we argued before each PPP_j is generally set to be some sort of averages of p_{ij} s deflated by international prices P_i s and each P_i is often set to be some sort of averages of p_{ij} s deflated by PPP_j s. Our equations imply that each PPP_j or P_i is in the form of generalized average of $g_i(P_i)\big/e_{ij}$ s and $e_{ij}\, f_j(PPP_j)$ with weights (i.e. $d_{ij}\Big/\sum_{i=1}^{N} d_{ij}$ and $c_{ij}\Big/\sum_{i=1}^{N} c_{ij}$) that add up to one. Here e_{ij} plays the role of p_{ij} s but we can allow it to be anything that is positive and the theorem still holds.

(ii) It appears that the weights cannot be completely independent and a condition such as $c_{ij}=d_{ij}e_{ij}$ is required to guarantee existence of a unique positive solution. For instance, consider the following system

$$\begin{cases} \dfrac{1}{PPP_j}=\sum_{i=1}^{N}\left(w_{ij}\,\dfrac{P_i}{p_{ij}}\right) & (j=1,.....,M) \\ P_i=\sum_{j=1}^{M}\left(w_{ij}^{*}\,\dfrac{p_{ij}}{PPP_j}\right) & (i=1,.....,N) \end{cases}$$

where PPP_j equations are the same as those in GK and Iklé index and P_i equations are the same as those in arithmetic index. This system satisfies all the conditions except $c_{ij}=d_{ij}e_{ij}$.

One can easily construct examples for which the system does not have a solution[15]. Another example is a system studied in Khamis and Rao (1989) which satisfies all the conditions except the particular form assumed for c_{ij}. They show that the system has only a trivial solution.

(iii) Second, equations (15.a and 15.b) are in terms of functions of PPP_j and P_i so that the functions involved can be written in the form of linear equations in suitably defined functions of PPP_j and P_i. For example, consider the Geary-Khamis system in equation (5)

$$\begin{cases} PPP_j = \sum_{i=1}^{N} p_{ij} q_{ij} \Big/ \sum_{i=1}^{N} P_i q_{ij} & (j = 1, \ldots\ldots, M) \\ P_i = \sum_{j=1}^{M} \frac{q_{ij}}{\sum_{j=1}^{M} q_{ij}} \frac{p_{ij}}{PPP_j} & (i = 1, \ldots\ldots, N) \end{cases}$$

The first equation for PPP_j is a nonlinear function of P_i's. However, this can be written as a linear system in $f(PPP_j) = 1/PPP_j$ and $g(P_i) = P_i$ as:

$$\begin{cases} f\left(PPP_j\right) = \sum_{i=1}^{N} \frac{p_{ij} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}} \frac{g\left(P_i\right)}{p_{ij}} & (j = 1, \ldots\ldots, M) \\ g\left(P_i\right) = \sum_{j=1}^{M} \frac{q_{ij}}{\sum_{j=1}^{M} q_{ij}} p_{ij} f(PPP_j) & (i = 1, \ldots\ldots, N) \end{cases}$$

where the functions a_{ij} and b_{ij} are suitably defined with $e_{ij} = p_{ij}$.

(iv) We note that not all multilateral systems can be written in the form of equations (15.a and 15.b).

(v) We can rewrite the system into the following equivalent form which is more convenient and we use this form next

[15] - For example the two- country-two-commodity case with $w_{11} = w_{12} = w_{21} = w_{22} = 0.5$ and $p_{11} = 0.5$, $p_{12} = p_{21} = p_{22} = 1$.

$$
\begin{cases}
f_j\left(PPP_j\right) = \sum_{i=1}^{N} \dfrac{d_{ij}(\mathbf{p},\mathbf{q})}{\sum_{i=1}^{N} c_{ij}(\mathbf{p},\mathbf{q})} g(P_i) & (i = 1,\dots,N) \\
g_i\,(P_i) = \sum_{j=1}^{M} \dfrac{c_{ij}(\mathbf{p},\mathbf{q})}{\sum_{j=1}^{M} d_{ij}(\mathbf{p},\mathbf{q})} f_j(PPP_j) & (j = 1,\dots,M)
\end{cases}
$$

Proof of Theorem 1:

To prove this Theorem, we establish the following two lemmas and to simplify exposition, we use $f_j = f(PPP_j)$ and $g_i = g(P_i)$. If we obtain solutions for f_j and g_i then we can obtain solutions for PPP_j and P_i by invoking the bijective nature of these functions.

Lemma 1: Consider the following system of equations defined in terms of (**f, g**)

$$
\begin{cases}
f_j = \sum_{i=1}^{N} \dfrac{d_{ij}}{\sum_{i=1}^{N} c_{ij}} g_i & (i = 1,\dots..,N) & \text{(16.a)} \\
g_i = \sum_{j=1}^{M} \dfrac{c_{ij}}{\sum_{j=1}^{M} d_{ij}} f_j & (j = 1,\dots..,M) & \text{(16.b)}
\end{cases}
$$

then a necessary and sufficient condition for existence of a unique positive $\mathbf{f}^* = (f_1^*,\dots f_M^*)$ and $\mathbf{g}^* = (g_1^*,\dots g_N^*)$ (up to a positive scalar factor) is indecomposability of matrix **B**, **C** or **D** defined below.

Proof of Lemma1: Through direct substitution, we first express the system (16.a) and (16.b) in the following matrix form:

$$
\mathbf{BX=X} \qquad \text{or} \qquad \begin{pmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{D} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{X_1} \\ \mathbf{X}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X_1} \\ \mathbf{X}_2 \end{pmatrix}.
$$

where **B**, **C** and **D** can be identified from the expression below:

$$\begin{bmatrix} 0 & \cdots & 0 & \frac{c_{11}}{\sum_{i=1}^{N} c_{i1}} & \cdots & \frac{c_{1M}}{\sum_{i=1}^{N} c_{iM}} \\ \vdots & & \vdots & \vdots & & \\ 0 & \cdots & 0 & \frac{c_{N1}}{\sum_{i=1}^{N} c_{i1}} & \cdots & \frac{c_{NM}}{\sum_{i=1}^{N} c_{iM}} \\ \frac{d_{11}}{\sum_{j=1}^{M} d_{1j}} & \cdots & \frac{d_{N1}}{\sum_{j=1}^{M} d_{Nj}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{d_{1M}}{\sum_{j=1}^{M} d_{1j}} & \cdots & \frac{d_{NM}}{\sum_{j=1}^{M} d_{Nj}} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} g_1 \sum_{j=1}^{M} d_{1j} \\ \vdots \\ g_N \sum_{j=1}^{M} d_{Nj} \\ f_1 \sum_{i=1}^{N} c_{i1} \\ \vdots \\ f_M \sum_{i=1}^{N} c_{iM} \end{bmatrix} = \begin{bmatrix} g_1 \sum_{j=1}^{M} d_{1j} \\ \vdots \\ g_N \sum_{j=1}^{M} d_{Nj} \\ f_1 \sum_{i=1}^{N} c_{i1} \\ \vdots \\ f_M \sum_{i=1}^{N} c_{iM} \end{bmatrix}$$

Defining **E=CD** and **F=DC** we can focus on the subsystem involving only $\mathbf{f}^* = (f_1^*, \ldots f_M^*)$ which is given by:

$$\mathbf{FX}_2 = \mathbf{X}_2 \Rightarrow \begin{bmatrix} \frac{\sum_{i=1}^{N} \frac{d_{i1} c_{i1}}{\sum_{j=1}^{M} d_{ij}}}{\sum_{i=1}^{N} c_{i1}} & \cdots & \frac{\sum_{i=1}^{N} \frac{d_{i1} c_{iM}}{\sum_{j=1}^{M} d_{ij}}}{\sum_{i=1}^{N} c_{iM}} \\ \vdots & \ddots & \vdots \\ \frac{\sum_{i=1}^{N} \frac{d_{iM} c_{i1}}{\sum_{j=1}^{M} d_{ij}}}{\sum_{i=1}^{N} c_{i1}} & \cdots & \frac{\sum_{i=1}^{N} \frac{d_{iM} c_{iM}}{\sum_{j=1}^{M} d_{ij}}}{\sum_{i=1}^{N} c_{iM}} \end{bmatrix} \begin{bmatrix} f_1 \sum_{i=1}^{N} c_{i1} \\ \vdots \\ \vdots \\ f_M \sum_{i=1}^{N} c_{iM} \end{bmatrix} = \begin{bmatrix} f_1 \sum_{i=1}^{N} c_{i1} \\ \vdots \\ \vdots \\ f_M \sum_{i=1}^{N} c_{iM} \end{bmatrix} \qquad (17)$$

or the subsystem involving $\mathbf{g}^* = (g_1^*, \ldots g_N^*)$ as:

$$\mathbf{EX}_1 = \mathbf{X}_1 = \begin{bmatrix} \dfrac{\sum_{j=1}^{M} \dfrac{d_{1j} c_{1j}}{\sum_{i=1}^{N} c_{ij}}}{\sum_{j=1}^{M} d_{1j}} & \cdots & \dfrac{\sum_{j=1}^{M} \dfrac{d_{Nj} c_{1j}}{\sum_{i=1}^{N} c_{ij}}}{\sum_{j=1}^{M} d_{Nj}} \\ \vdots & \ddots & \vdots \\ \dfrac{\sum_{j=1}^{M} \dfrac{d_{1j} c_{Nj}}{\sum_{i=1}^{N} c_{ij}}}{\sum_{j=1}^{M} d_{1j}} & \cdots & \dfrac{\sum_{j=1}^{M} \dfrac{d_{Nj} c_{Nj}}{\sum_{i=1}^{N} c_{ij}}}{\sum_{j=1}^{M} d_{Nj}} \end{bmatrix} \begin{bmatrix} g_1 \sum_{j=1}^{M} d_{1j} \\ \vdots \\ \vdots \\ g_N \sum_{j=1}^{M} d_{Nj} \end{bmatrix} = \begin{bmatrix} g_1 \sum_{j=1}^{M} d_{1j} \\ \vdots \\ \vdots \\ g_N \sum_{j=1}^{M} d_{Nj} \end{bmatrix}$$

These systems have been written in a way to ensure each columns sum to one. We can write each of these linear systems as $\mathbf{AX} = \mathbf{X}$ where matrix **A** (i.e. **B** or **E** or **F**) is a non-negative matrix with all columns sums to one.

To prove necessity part of the theorem, we note that the matrix $\mathbf{A}'$, transpose of $\mathbf{A}$, is a stochastic matrix[16] (column sums of matrix $\mathbf{A}$ equals 1) therefore it has a dominant eigenvalue equal to one with corresponding eigenvector equal to (1,…,1) [see e.g. Gantmacher 1959, pp 100]. Now for $\mathbf{AX} = \mathbf{X}$ to have a unique positive solution, matrix **A** must have a dominant eigenvalue equal to one with a corresponding positive eigenvector. But this (i.e. both **A** and $\mathbf{A}'$ having the same dominant eigenvalue with positive eigenvectors) is possible only if **A** is indecomposable (see corollary in page 96 of Gantmacher 1959).

To prove sufficiency we use the Frobenius-Perron theorem. According to Perron-Frobenius theorem if matrix **A** is indecomposable then $\mathbf{AX} = \lambda \mathbf{X}$ has a unique positive solution with $\lambda > 0$. Furthermore since each column of **A** sums to one, we must have $\lambda = 1$. Note also that properties of the $\mathbf{q}$ matrix ensures that $\sum_{j=1}^{M} d_{ij} > 0$ and $\sum_{i=1}^{N} c_{ij} > 0$ therefore $\mathbf{f}^* = (f_1^*, \ldots f_M^*)$ and $\mathbf{g}^* = (g_1^*, \ldots g_N^*)$ are well-defined[17].

[16] A matrix is said to be stochastic if it is non-negative and each row sums to one.

[17] Note that connectedness is necessary for uniqueness of the solutions. Without connectedness there can be solutions but not unique since the system of equations for some $J \subset \{1, \ldots, M\}$ can be divided at least into two independent subsystems with independent solutions $\mathbf{f}_J^* = \{ f_j^* \mid j \in J \}$ and $\mathbf{f}_{J^c}^* = \{ f_j^* \mid j \in J^c \}$ with $\gamma_1 \mathbf{f}_J^*$ and $\gamma_2 \mathbf{f}_{J^c}^*$ for any $\gamma_1 > 0$, $\gamma_2 > 0$ as solutions which violates the uniqueness.

Lemma 2: A necessary and sufficient condition for matrix **E** in (17), and therefore **B** and **F**, to be indecomposable is the connectedness of the set of countries based on the quantity matrix **q**.

Sufficiency: If **q** is connected, then for any non-empty subset $J \subset \{1, 2, ..., M\}$, there exists at least one $m \in J$ and $k \in \{1, ..., N\}$ such that $q_{km} > 0$ and at least one $l \notin J$ such that $q_{kl} > 0$. Note that this implies $d_{kl} > 0$, $c_{kl} > 0, d_{km} > 0$, $c_{km} > 0$. The sufficiency is proved if we show that $\{\mathbf{E}\}_{ml}$ (i.e. element $\{m, l\}$ of the matrix **E**) is positive. Note that

$$\{\mathbf{E}\}_{ml} = \frac{\sum_{i=1}^{N} d_{im} c_{il} \Big/ \sum_{j=1}^{M} d_{ij}}{\sum_{i=1}^{N} c_{il}}$$

It can be seen that this is positive if and only if there is at least one $k \in \{1, ..., N\}$ such that both $d_{km} > 0$ *and* $c_{kl} > 0$ but connectedness as argued in the previous paragraph guarantees existence of such a *k*.

Necessity: If matrix **E** is indecomposable then for any proper subset $J \subset \{1, 2, ..., M\} \neq \varnothing$ there exists at least one $m \in J$ and $l \notin J$ such that $\{\mathbf{E}\}_{ml} > 0$ which implies existence of at least one $k \in \{1, ..., N\}$ such that $d_{km} > 0$ and $c_{kl} > 0$ but this in turn implies existence of $q_{km} > 0$ and $q_{kl} > 0$ and therefore connectedness. Q.E.D

Proof of Theorem 1: From Lemma 1, indecomposability of matrices **B**, **C** or **D** is necessary and sufficient for the existence of a unique positive solution for the system (15a and 15b) in Theorem 1. Lemma 2 shows that connectedness of the quantity matrix **q** is necessary and sufficient for indecomposability of the matrices involved thus establishing Theorem 1.

From definitions 2, 3 and 4, we can translate the necessary and sufficient conditions on the elements of the quantity matrix **q** stated in Theorem 1 and restate it as: unique positive solutions to the multilateral systems of the form (15a and 15b) exist if and only the quantity-adjacent graph $G_{\mathbf{q}}$ is connected.

***Choice of functional forms for* $\mathbf{f}(.)$ *and* $\mathbf{g}(.)$**

In the lemmas and the theorem proved above, we have shown that if there is a positive $\mathbf{f} = (f_1, ... f_M)$ and $\mathbf{g} = (g_1, ... g_N)$ that solves the system of equations then $\delta\mathbf{f}$ and $\delta\mathbf{g}$ for every $\delta > 0$ is also a solution. We have assumed that f_j and g_i s are invertible functions, therefore there exist vectors $\mathbf{PPP}^*$ and $\mathbf{P}^*$ that solve systems (15.a) and (15.b). However in general if $\{PPP_j^*\}$ and $\{P_i^*\}$ is a solution then $\{f^{-1}[\delta f(PPP_j^*)]\}$ and $\{g^{-1}[\delta g(P_i^*)]\}$ is also a solution.

In order to narrow the class of functions $\mathbf{f}(.)$ and $\mathbf{g}(.)$, we invoke the following reasonable axiom which states that if the unit of measurement of the reference currency unit is multiplied by γ then the international average price of the commodity must be divided by γ. For example, if the reference currency is 1 US dollar and if the international average price of wheat per tonne is US$125; then when the reference currency is changed to 100 US dollars then the international average price would be 1.25 units of reference currency (US$100).

Axiom of Units of Reference Currency Unit: This axiom simply states that if $\mathbf{PPP}^*$ and $\mathbf{P}^*$ are solutions to the multilateral system, then $\gamma\mathbf{PPP}^*$ and $(1/\gamma)\mathbf{P}^*$ (for every $\gamma > 0$) is also a solution.

This axiom helps us to narrow the class of functions *f(.)* and *g(.)* that can be used in international comparisons. Lemma 3 below gives conditions on **f** and **g** that ensure this property.

Lemma 3: In order to have the axiom of change of reference currency unit, that is for every $\gamma > 0$ and $[\mathbf{PPP}, \mathbf{P}] > \mathbf{0}$, if $[\mathbf{PPP}, \mathbf{P}]$ is a solution to the system (15.a) and (15.b) $\Rightarrow$ $[\gamma\mathbf{PPP}, (1/\gamma)\mathbf{P}]$ is also a solution, it is both necessary and sufficient that $f(.)$ and $g(.)$ is of the form $f(x) = \alpha x^{\rho}$ and $g(y) = \beta y^{-\rho}$ for any $\alpha, \beta > 0$ and $\rho \in R$.[18]

Necessity: Note that having both $[\mathbf{PPP}, \mathbf{P}]$ and $[\gamma\mathbf{PPP}, (1/\gamma)\mathbf{P}]$ as solutions for any $\gamma > 0$ is equivalent to having both $[f(\mathbf{PPP}), g(\mathbf{P})]$ and $[f(\gamma\mathbf{PPP}), g(\frac{1}{\gamma}\mathbf{P})]$ as solutions. On the other hand, according to lemma-1 if $[f(\mathbf{PPP}), g(\mathbf{P})]$ is a solution then $[\delta f(\mathbf{PPP}), \delta g(\mathbf{P})]$ is also a solution for every $\delta > 0$. Therefore for every $\gamma > 0$ there exists some $\delta > 0$ so that:

[18] - We are not considering trivial situations where either $f_j(x) = 0$ or $g_i(y) = 0$.

$$\begin{cases} f_j(\gamma PPP_j) = \delta f_j(PPP_j) & \text{for } j = 1,2,...,M \\ g_i(\frac{1}{\gamma} P_i) = \delta g_i(P_i) & \text{for } i = 1,2,...,N \end{cases}$$

Since we are assuming this to be true for any possible solution $[\mathbf{PPP}, \mathbf{P}] > \mathbf{0}$, we can let $PPP_j = 1$ and $P_i = 1$ (e.g. when all prices in all countries are the same) and have

$$\begin{cases} f_j(\gamma) = \delta f_j(1) & \text{for } j = 1,2,...,M \\ g_i(1/\gamma) = \delta g_i(1) & \text{for } i = 1,2,...,N \end{cases}$$

Since this is true for any $\gamma > 0$

$$\begin{cases} f_j(\gamma) = f(\gamma) \quad \& \quad \delta = \dfrac{f(\gamma)}{f(1)} \\ g_i(1/\gamma) = g(1/\gamma) \quad \& \quad \delta = \dfrac{g(1/\gamma)}{f(1)} \end{cases}$$

then we can write the equations as

$$\begin{cases} f(\gamma PPP_j) = \dfrac{f(\gamma)}{f(1)} f(PPP_j) & \text{for } j = 1,2,...,M \\ g(\frac{1}{\gamma} P_i) = \dfrac{g(1/\gamma)}{f(1)} g(P_i) & \text{for } i = 1,2,...,N \end{cases}$$

where $f(\gamma)$ and $g(1/\gamma)$ are continuous functions defined over R_+. In general, each equation in the above system is a special form of 4th Pexider's functional equation (see e.g. Aczél 1966 or Diewert 2011). The non-trivial solution to this system of functional equations takes the form $f(x) = ax^{\rho_1}$ and $g(y) = \beta y^{\rho_2}$ for any $\rho_1, \rho_2 \in R$ and $f(1) = \alpha$ and $g(1) = \beta$. But note that for every $\gamma > 0$ we must have

$$\delta = \frac{f(\gamma)}{f(1)} = \frac{g(1/\gamma)}{g(1)} \Rightarrow \gamma^{\rho_1} = \gamma^{-\rho_2} \Rightarrow \rho_1 = -\rho_2$$

Sufficiency: The proof is trivial. Q.E.D

In the following *Corollary* we establish necessary and sufficient conditions for the existence of solutions to the Geary-Khamis system (5), the generalized *GK* system (6) and the generalize system in (7).

Corollary 1: A necessary and sufficient condition for existence and uniqueness of Geary-Khamis index (5), generalized additive index (6) and system (7) is the connectedness of quantity matrix $\mathbf{q}$.

In system (15.a) and (15.b) defining $f(x)=\frac{1}{x}$, $g(x)=x$, $c_{ij}=p_{ij}q_{ij}$, $d_{ij}=q_{ij}$ and $e_{ij}=p_{ij}$ leads to 5. This specification of f and g also ensure that the resulting **PPP** and **P** satisfy the property that for every $\gamma>0$ both $[\mathbf{PPP}^*,\mathbf{P}^*]$ and $[\gamma\mathbf{PPP}^*,\frac{1}{\gamma}\mathbf{P}^*]$ are solutions to the system (15.a) and (15.b). Existence and uniqueness of generalized additive index (6) can be proved by defining $c_{ij}=\beta_j p_{ij}q_{ij}$, $d_{ij}=\beta_j q_{ij}$. For system (7) define $f(x)=\frac{1}{x^{\rho}}$, $g(x)=x^{\rho}$ and $c_{ij}=\beta_j p_{ij}q_{ij}$ and $d_{ij}=\beta_j q_{ij}$.

Although Theorem (1) can be used for a variety of indexes, it doesn't cover all cases of interest. Now we state Theorem 2 which extends Theorem 1 to the case where d_{ij}s and c_{ij}s are functions of **P** as well. As we saw in Section 3, Rao (1976) and Neary (2004) offer examples where d_{ij} and c_{ij} are functions of elements of **P**. This general specification makes the system more complex and conditions for existence and uniqueness more difficult to prove. We can still provide the theorem in terms of functions of **P** and **PPP**s with conditions stated in Lemma 3. Without loss of generality we assume that $f(x)=1/x$ and $g(x)=x$, i.e. the value of ρ equal to 1.

Theorem 2: *Consider the following general system of equations*

$$\frac{1}{PPP_j}=\sum_{i=1}^{N}\frac{d_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}{\sum_{i=1}^{N}c_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}P_i \qquad (j=1,.....,M) \qquad (18.a)$$

$$P_i=\sum_{j=1}^{M}\frac{c_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}{\sum_{j=1}^{M}d_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}\frac{1}{PPP_j} \qquad (i=1,.....,N) \qquad (18.b)$$

*where $d_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})$ and $c_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})$ are positive continuous homogeneous functions of any degree with respect to **P**. Then the system has a non-negative solution. Furthermore if the vector function **G** defined below satisfies monotonicity and indecomposibility, the solution is positive and unique up to a positive scalar multiple.*

Proof: Substitute PPP_js in (18a) from (18.b) and define ***G*** as

$$G_i = \sum_{j=1}^{M} \frac{c_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}{\sum_{j=1}^{M} d_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})} \sum_{i=1}^{N} \frac{d_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})}{\sum_{i=1}^{N} c_{ij}(\mathbf{P},\mathbf{p},\mathbf{q})} P_i \qquad (i = 1,\ldots\ldots, N)$$

It is easy to see that ***G*** satisfies conditions (*i*), (*ii*) of *nonlinear eigenvalue theorem* stated in Section 2. Under these conditions there is at least one non-negative solution for the system (see e.g. theorem 10.2 pp 151 in Nikaido 1968). We need to establish that the eigenvalue associated with the solution is equal to 1. Since there is at least one solution we can write:

$$\lambda^* P_i^* = \sum_{j=1}^{M} \frac{c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})}{\sum_{j=1}^{M} d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})} \sum_{i=1}^{N} \frac{d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})}{\sum_{i=1}^{N} c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})} P_i^* \qquad (i = 1,\ldots\ldots, N)$$

After some manipulations we have

$$\lambda^* \sum_{j=1}^{M} d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) P_i^* = \sum_{j=1}^{M} c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) \sum_{i=1}^{N} \frac{d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})}{\sum_{i=1}^{N} c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})} P_i^* \qquad (i = 1,\ldots\ldots, N)$$

Summing equations over *i* we have

$$\lambda^* \sum_{i=1}^{N} \sum_{j=1}^{M} d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) P_i^* = \sum_{j=1}^{M} \frac{\sum_{i=1}^{N} c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) \left(\sum_{i=1}^{N} d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) P_i^* \right)}{\sum_{i=1}^{N} c_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q})} = \sum_{j=1}^{M} \sum_{i=1}^{N} d_{ij}(\mathbf{P}^*,\mathbf{p},\mathbf{q}) P_i^* \Rightarrow \lambda^* = 1$$

This completes the first part of the theorem establishing the existence of a positive solution to the system. In addition, if ***G*** also satisfies conditions (iii) and (iv) i.e. monotonicity and indecomposibility then the solution is unique which follows from nonlinear eigenvalue theorem. Q.E.D

Note that the system has always at least one non-negative solution but in order to prove a unique positive solution using nonlinear eigenvalue theorem we need monotonicity and indecomposibility. Indecomposibility is likely to be satisfied if the countries are connected and $q_{ij} > 0 \Rightarrow c_{ij} > 0 \Rightarrow d_{ij} > 0$. But without knowing the exact form of c_{ij} and d_{ij}s or stringent conditions on these functions we cannot prove monotonicity of the function **G**[19].

Corollary 2: The Neary system (13), have at least one non-negative solution.

In system (15) define $c_{ij} = p_{ij}q_{ij}$ and $d_{ij} = q_{ij}^*$ where it is evident that c_{ij}s are homogenous of degree zero and d_{ij}s are homogenous of degree one since q_{ij}^*s are demand functions.

We now focus on a variant of Theorem 1 which can be used in establishing existence and uniqueness of solutions for systems that are not covered by equation system 2.3. The new equation system in (19.a, 19.b) of Theorem 3 covers multilateral index number systems proposed in Rao (1990) and Iklé (1972) described in Section 3. To address such problems we need a different nonlinear Perron-Frobenius problem known as the DAD problem (see e.g. Chapter 7 of Lemmens and Nussbaum 2012 or Menon and Schneider 1969). The DAD problem involves a triplet $\{\mathbf{A}, \mathbf{c}, \mathbf{d}\}$ where $\mathbf{A} = \{a_{ij}\}$ is an $N \times M$ non-negative matrix, $\mathbf{c}$ is an $N \times 1$ positive vector and $\mathbf{d}$ is an $M \times 1$ positive vector where it is asked that under what conditions[20] the following system has a unique positive solution:

$$\sum_{i=1}^{N} \frac{a_{ik} c_i}{\sum_{j=1}^{M} \frac{a_{ij} d_j}{x_j}} = x_k \qquad \textit{for } k = 1, ..., M$$

It has been shown that a necessary and sufficient condition for existence of at least one positive solution is the compatibility condition (defined below). A further condition of connectedness (defined below) provides necessary and sufficient for uniqueness of the solution[21].

Compatibility condition: For every $I \subseteq \{1, ..., N\}$ and $J \subseteq \{1, ..., M\}$ define I^c and J^c as complement of these sets. Compatibility condition implies that for every $\mathbf{A}_{I^c J^c} = \mathbf{0}$, the inequality $\sum_{i \in I^c} c_i \le \sum_{j \in J} d_j$ holds, and inequality is strict if and only if $\mathbf{A}_{IJ} \ne \mathbf{0}$.

[19] Monotonicity has been proved in the case of Rao (1976) system, However, it may be difficult to show this in the case of Neary (2004) as it includes q* and q in the definitions.

[20] The DAD problem in its original form asks, for a square matrix **A** with nonnegative entries, when it is possible to find positive diagonal matrices $\mathbf{D_1}$ and $\mathbf{D_2}$ with $\mathbf{D}_1\mathbf{A}\mathbf{D}_2$ to have row sums equal to **d** and column sums equal to **c.**

[21] - Menon and Schneider 1969) do not mention necessity of connectedness for uniqueness but it is easy to see that it is necessary using exactly the same argument as in footnote 17.

Connectedness[22] ***of*** $\mathbf{A}$: For every $I \subset \{1,...,N\}$, $J \subset \{1,...,M\}$ if $\mathbf{A}_{I^c J^c} = \mathbf{0} \Rightarrow \mathbf{A}_{IJ} \neq \mathbf{0}$.

Theorem 3-1: *Suppose $f_j(.)$s and $g_j(.)$s are known non-negative, bijective functions defined over the non-negative values and*

$$\begin{cases} f_j(PPP_j) = \sum_{i=1}^{N} \dfrac{a_{ij} c_i}{g_i(P_i)} & (j = 1, ..., M) \qquad (19.a) \\ g_i(P_i) = \sum_{j=1}^{M} \left(\dfrac{a_{ij} d_j}{f_j(PPP_j)} \right) & (i = 1, ..., N) \qquad (19.a) \end{cases}$$

with, $b_j > 0$, $c_j > 0$ and $a_{ij} \geq 0$ the above compatilibity and connectedness conditions are necessary and sufficient for having a unique positive solution.

Proof: Substituting (19.b) into (19.a), we obtain

$$f_k = \sum_{i=1}^{N} \frac{a_{ik} c_i}{\sum_{j=1}^{M} \dfrac{a_{ij} d_j}{f_j}} \qquad (k = 1, ..., M)$$

But this is exactly the DAD problem discussed above where compatibility and indecomposibility conditions are sufficient for existence of a unique positive solution up to a positive scalar multiple.

The difference between this and (15.a and 15.b) is that (19.a and 19.b) cannot be turned into a linear system in terms of g_is and f_js and there are significant differences in the weights attached to $g_i(P_i)$ and $f_j(PPP_j)$ in these systems. In this general form, we cannot relate the solution of the system to connectedness of the countries. However, the more interesting special case of this system can be written as

$$\begin{cases} PPP_j = f_j^{-1} \left(\sum_{i=1}^{N} \dfrac{d_{ij}}{\sum_{i=1}^{N} d_{ij}} \dfrac{e_{ij}}{g_i(P_i)} \right) & (j = 1, ..., M) \\ P_i = g_i^{-1} \left(\sum_{j=1}^{M} \dfrac{d_{ij}}{\sum_{j=1}^{M} d_{ij}} \dfrac{e_{ij}}{f_j(PPP_j)} \right) & (i = 1, ..., N) \end{cases}$$

[22] Menon and Schneider 1969) call this condition indecomposibility.

with $e_{ij} > 0 \ and \, q_{ij} > 0 \Leftrightarrow d_{ij} > 0$. We can show that connectedness of countries is a sufficient condition for existence of a unique positive solution and also systems (9,10,11,12) all can be written in this form.

Theorem 3-2: Consider the following system of equations

$$\begin{cases} f_j = \sum_{i=1}^{N} \frac{d_{ij}}{\sum_{i=1}^{N} d_{ij}} \frac{e_{ij}}{g_i} & (j = 1, \ldots.., M) \quad (20.a) \\ g_i = \sum_{j=1}^{M} \frac{d_{ij}}{\sum_{j=1}^{M} d_{ij}} \frac{e_{ij}}{f_j} & (i = 1, \ldots.., N) \quad (20.b) \end{cases}$$

Then connectedness of countries as in Definition 2 is both necessary and sufficient for uniqueness of the solutions (up to a positive scalar multiple).

Proof: Note first that this problem is a special case of (19.a) and (19.b) by defining $a_{ij} = \frac{d_{ij} e_{ij}}{\sum_{j=1}^{M} d_{ij} \sum_{i=1}^{N} d_{ij}}$, $c_i = \sum_{j=1}^{M} d_{ij}$ and $d_j = \sum_{i=1}^{N} d_{ij}$

Sufficiency: Define $\mathbf{D} = \{d_{ij} \mid i = 1, \ldots, N, j = 1, \ldots, M\}$ then connectedness of $\mathbf{q}$ $\Leftrightarrow$ connectedness of $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{D}$ since $d_{ij} > 0 \Leftrightarrow q_{ij} > 0$. The next step is to show that connectedness of $\mathbf{D}$ is sufficient for compatibility.

Let $I \subset \{1, \ldots, N\}$, $J \subset \{1, \ldots, M\}$ and let $\mathbf{A}_{I^c J^c} = \mathbf{0}$ then

$$\sum_{i \in I^c} c_i = \sum_{i \in I^c} \sum_{j \in J \cup J^c} d_{ij} = \sum_{i \in I^c} \sum_{j \in J} d_{ij} \text{ since } \sum_{i \in I^c} \sum_{j \in J^c} d_{ij} = 0$$

but we have

$$\sum_{j \in J} d_j = \sum_{i \in I^c \cup I} \sum_{j \in J} d_{ij} > \sum_{i \in I^c} \sum_{j \in J} d_{ij} > \sum_{i \in I^c} c_i$$

since connectedness implies that for at least one $i \in I$ and $j \in J^c$ there must be a $d_{ij} > 0$,if not, then the set of countries can be divided into two disjoint groups with no commodities in common (group J would only have commodities belonging to I^c and group J^c have only commodities belonging to I) which violates the connectedness assumption.

Necessity: If there is a unique psitive solution to (20.a) and (20.b), $\mathbf{A} = \{a_{ij} \mid i = 1,\ldots,N,\ j = 1,\ldots,M\}$ must be connected and connectedness of $\mathbf{A}$ $\Rightarrow$ connectedness of **q** due to the assumption $d_{ij} > 0 \Leftrightarrow q_{ij} > 0$.

Q.E.D

If the condition that for every $\gamma > 0$ both $[\mathbf{PPP}^*, \mathbf{P}^*]$ and $[\gamma \mathbf{PPP}^*, (1/\gamma)\mathbf{P}^*]$ as solutions to the system (19.a) and (19.b) is imposed, then using arguments similar to lemma 3 above the functional forms used in the system must be set as $f_j(x) = \alpha x^{\rho}$ and $g_i(x) = \beta x^{\rho}$ for any $\alpha > 0, \beta > 0$ and $\rho \in R$.

Corollary 3: If matrix **q** is connected. Systems (9), (10), (11) and (12) have unique positive solutions.

We consider system (12) which encompasses all the others. Note that if in (20.a) and (20.b) we define $f_j(x) = g_i(x) = x^{\rho}$ and $d_{ij} = w_{ij}, e_{ij} = p_{ij}^{\rho}$, $c^i = \sum_{j=1}^{M} w_{ij}$ and $d_j = \sum_{i=1}^{N} w_{ij} = 1$ we obtain system (12) and therefore theorem (3.2) proves its existence and uniqueness.

5. Conclusions

In this paper we focus on the existence and uniqueness of a number of multilateral index number systems used in the literature. The main objective of the paper is to develop a toolkit and prove a number of existence theorems which can be used in establishing the existence of solutions to a large array of multilateral systems used widely in the context of international comparisons of prices and real incomes. We use the well-known nonlinear eigenvalue theorem, the Frobenius-Perron theorem and a slightly different version of the nonlinear eigenvalue theorem known as the DAD. The main conclusion emanating from the conditions that underpin various theorems in the paper is that a necessary and sufficient condition for the existence of solutions is that the observed quantities of different commodities in different countries and therefore the associated graph are connected. While simple connectedness guarantees the existence of solutions and therefore the viability of most of the multilateral index number systems, the strength of connectedness could have implications for the reliability of the models. The theorems proved in this paper are general and powerful enough to prove the existence and uniqueness of solutions not only to the currently used system of index numbers that make use of the twin concepts of international average prices and purchasing power parities but also systems that may come into vogue in the future.

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